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The Structure of the Solution Sets of Zero-Sum Games

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The solution sets of finite matrix games are thoroughly studied by many authors. In this paper for bounded semi-infinite matrix games the structure of the optimal strategy spaces is investigated. It is proved that for a bounded $m\times\infty$-matrix game A, the solution set $O_I(A)$ of player I is a non-empty compact convex subset of the set S^m of all mixed strategies; and, conversely, that each non-empty compact convex subset of S^m is a solution set of player I for some $m\times\infty$-matrix game. The solution set $O_{II}(A)$ of player II is a closed convex subset of l^1 with a countable number of extreme points. These extreme points are associated with certain square submatrices of A. Also some characterizing properties of the set $O_{II}^c(A)$ of optimal strategies of player II with a finite carrier are given.

1. INTRODUCTION

Let $m \in \mathbb{N}$, $n \in \mathbb{N} \cup \{\infty\}$ and let $A = [a_{ij}]_{i=1,j=1}^{m\ \ n}$ be a bounded $m\times n$-matrix of real numbers. Let

$$S^m := \{p \in \mathbb{R}^m;\ p \geq 0,\ \Sigma_{i=1}^m p_i = 1\} \text{ for each } m \in \mathbb{N},$$

$$S^\infty := \{q \in l^1;\ q \geq 0,\ \Sigma_{j=1}^\infty q_j = 1\},$$

where l^1 is the normed linear space, consisting of those infinite sequences $x = (x_1, x_2, \ldots)$ of real numbers for which $\|x\|_1 := \Sigma_{i=1}^\infty |x_i| < \infty$. It is well-known that for the matrix game corresponding with A, the lower value $\sup_{p\in S^m} \inf_{q\in S^n} pAq^t$ is equal to the upper value $\inf_{q\in S^n} \sup_{p\in S^m} pAq^t$; this common value will be denoted by $v(A)$. For finite matrix games ($n \in \mathbb{N}$) the existence of a value was proved already in 1928 by J. von Neumann [7]; the existence of a value for bounded semi-infinite matrix games ($n = \infty$) was proved for the first time by A. Wald in [10]. The sets

$$O_I(A) := \{p \in S^m;\ pAq^t \geq v(A) \text{ for each } q \in S^n\},$$

$$O_{II}(A) := \{q \in S^n;\ pAq^t \leq v(A) \text{ for each } p \in S^m\}$$

are called the *solution sets or optimal strategy spaces of player* I

and player II respectively. In the papers [8], [4] and [2], which appeared in the "Contributions to the Theory of Games, volume 1" (Princeton University Press) much attention is paid to the structure of $O_I(A)$ and $O_{II}(A)$ for the case that A is a finite matrix. $O_I(A)$ and $O_{II}(A)$ appeared to be non-empty compact convex subsets of S^m and S^n respectively, both with a finite number of extreme points. In [8] Shapley and Snow associated these extreme points with certain square submatrices of A. In the papers [4] and [2] the following question was solved: Which pairs C_1 and C_2 of closed convex polyhedra can serve as pairs of optimal strategy spaces for some finite matrix game?
It is the purpose of this paper to study the structure of the optimal strategy spaces $O_I(A)$ and $O_{II}(A)$ for bounded $m\times\infty$-matrix games A.

2. PROPERTIES OF THE OPTIMAL STRATEGY SPACES OF SEMI-INFINITE MATRIX GAMES

In the following $A = [a_{ij}]_{i=1,j=1}^{m\ \ \infty}$ is a bounded $m\times\infty$-matrix ($m \in \mathbb{N}$). We start with two theorems about the optimal strategy space of player I.

THEOREM 1. $O_I(A)$ *is a non-empty compact convex subset of* S^m.

PROOF. Let $f : S^m \to \mathbb{R}$ be the function defined by

$$f(p) := \inf_{q\in S^\infty} pAq^t \text{ for each } p \in S^m.$$

Then f is an upper semi-continuous concave function on S^m because f is the infimum of continuous affine functions of the form $p \mapsto pAq^t$ ($p\in S^m$). Since S^m is compact, the subset of S^m where f is maximal is a non-empty closed convex set, and this subset coincides with $O_I(A)$. $\|$

Now we show that each non-empty compact convex subset of S^m is an optimal strategy space of player I for some $m\times\infty$-matrix game.

THEOREM 2. *Let* C *be a non-empty compact convex subset of* S^m. *Then there exists a bounded* $m\times\infty$*-matrix* A *such that* $v(A) = 0$ *and* $O_I(A) = C$.

PROOF. Take a bounded sequence $n_1, n_2, \ldots$ in $\mathbb{R}^m \setminus \{0\}$ such that

$$C = S^m \cap \bigcap_{j=1}^{\infty} \{x \in \mathbb{R}^m ;\ xn_j^t \geq 0\}.$$

Let A be the bounded $m\times\infty$-matrix with columns $0^t, n_1^t, n_2^t, \ldots$
From $pA \geq (0,0,\ldots)$ for each $p \in C$ and $Ae_1^t = (0,0,\ldots,0)^t$ we may con-

clude that $v(A) = 0$, $C \subset O_I(A)$ and $\{e_1\} \subset O_{II}(A)$. Now let $x \in S^m \setminus C$. Then we can find a $j \in \mathbb{N}$ such that $xn_j^t < 0 = v(A)$. This implies that $xAe_{j+1}^t < v(A)$ and so $x \notin O_I(A)$. Thus we have proved that $O_I(A) = C$. ||

In the following the subset of l^1 consisting of those elements $q = (q_1, q_2, \ldots)$, for which there is an $n \in \mathbb{N}$ such that $q_j = 0$ for each $j > n$, is denoted by l_c^1. Further $S^c := S^\infty \cap l_c^1$ and $O_{II}^c(A) := O_{II}(A) \cap l_c^1$. The convex hull of a set P in a linear space is denoted by ch(P) and the set of extreme points of a convex set C is denoted by ext(C).
The closure of a subset C of l^1 is denoted by cl(C).

The relationship between $O_{II}(A)$ and $O_{II}^c(A)$ is studied in the theorems 3 and 4 with the aid of the following lemma. For a proof of this lemma, see [1] p.48. (Cf. [5] p.201, excercise 3.70.)

<u>LEMMA 1</u>. *Let $v_1, v_2, \ldots$ be an infinite sequence of elements in $\mathbb{R}^m$. Let $q \in S^\infty$ be such that $\Sigma_{j=1}^\infty q_j v_j$ is an element of $\mathbb{R}^m$. Then there is a $\hat{q} \in S^c$ such that*

$$\Sigma_{j=1}^\infty q_j v_j = \Sigma_{j=1}^\infty \hat{q}_j v_j \in ch\{v_1, v_2, \ldots\}.$$

<u>THEOREM 3</u>. $O_{II}(A) = cl(O_{II}^c(A))$.

<u>PROOF</u>. (a) Let $q \in O_{II}(A)$. We show that $O_{II}(A) \subset cl(O_{II}^c(A))$ by proving that for each $\varepsilon > 0$ there is a $q_\varepsilon \in O_{II}^c(A)$ such that $\|q-q_\varepsilon\|_1 < \varepsilon$. Let $\varepsilon > 0$. Take an $N \in \mathbb{N}$ such that $t := \Sigma_{j=N+1}^\infty q_j < \frac{1}{2}\varepsilon$. If $t = 0$, let $q_\varepsilon := q$ and we have $\|q-q_\varepsilon\|_1 = 0 < \varepsilon$. Suppose that $t > 0$.
Let $a := \Sigma_{j=1}^\infty q_j Ae_j^t \leq (v(A), \ldots, v(A))^t$. Then $a \in (\mathbb{R}^m)^t$. Put $v_j := Ae_{N+j}^t$ for each $j \in \mathbb{N}$ and $r := (t^{-1}q_{N+1}, t^{-1}q_{N+2}, \ldots) \in S^\infty$. Then

$$\Sigma_{j=1}^\infty r_j v_j = t^{-1}(a - \Sigma_{j=1}^N q_j Ae_j^t) \in (\mathbb{R}^m)^t .$$

By applying lemma 1 (with r in the role of q) we get

$$t^{-1}(a - \Sigma_{j=1}^N q_j Ae_j^t) \in ch\{Ae_{N+1}^t, Ae_{N+2}^t, \ldots\}.$$

Let $(s_{N+1}, s_{N+2}, \ldots) \in S^c$ be such that

$$t^{-1}(a - \Sigma_{j=1}^N q_j Ae_j^t) = \Sigma_{j=N+1}^\infty s_j Ae_j^t.$$

Let $q_\varepsilon := (q_1, q_2, \ldots, q_N, ts_{N+1}, ts_{N+2}, \ldots)$. Then $q_\varepsilon \in S^c$, $\|q_\varepsilon - q\|_1 < \varepsilon$

and

$$Aq_\varepsilon^t = \Sigma_{j=1}^{N} q_j Ae_j^t + \Sigma_{j=N+1}^{\infty} ts_j Ae_j^t = a \leq (v(A), v(A), \dots, v(A))^t.$$

Hence $q_\varepsilon \in O_{II}^c(A)$.

(b) Now we show that $cl(O_{II}^c(A)) \subset O_{II}(A)$. Let $q^1, q^2, \dots$ be a sequence in $O_{II}^c(A)$ converging to some $q \in l^1$. Then $q \in S^\infty$. For each $i \in \{1,\dots,m\}$, the function $x \mapsto e_i Ax^t$ $(x \in S^\infty)$ is a continuous function on S^∞ and $e_i A(q^k)^t \leq v(A)$ for each $k \in \mathbb{N}$. Thus, we may conclude that $e_i Aq^t \leq v(A)$ for each $i \in \{1,\dots,m\}$. So $q \in O_{II}(A)$. $\|$

<u>THEOREM 4</u>. $ext(O_{II}(A)) = ext(O_{II}^c(A))$.

<u>PROOF</u>. (a) First we prove that $ext(O_{II}(A)) \subset O_{II}^c(A)$. Take a $q \in O_{II}(A)$, $q \notin S^c$. In view of lemma 1 there is an $r \in S^c$ such that $Aq^t = Ar^t$ and such that $r_j = 0$ if $q_j = 0$. Therefore we can take an $\varepsilon > 0$, sufficiently small, such that $z := (1+\varepsilon)q - \varepsilon r \in S^\infty$. Then $q = (1+\varepsilon)^{-1} z + (1+\varepsilon)^{-1} \varepsilon r \in ch\{z,r\}$ and $z \neq r$, because $r \neq q$. But this implies that $q \notin ext(O_{II}(A))$ since $z, r \in O_{II}(A)$. Hence $ext(O_{II}(A)) \subset S^c$, and so $ext(O_{II}(A)) \subset O_{II}^c(A)$.

(b) Because $O_{II}^c(A) \subset O_{II}(A)$, it follows from (a) that

$$ext(O_{II}(A)) = ext(O_{II}(A)) \cap O_{II}^c(A) \subset ext(O_{II}^c(A)).$$

(c) Now we prove that $ext(O_{II}^c(A)) \subset ext(O_{II}(A))$. Take a $q \in ext(O_{II}^c(A))$. Suppose that $q = \frac{1}{2}(q^1+q^2)$ for some $q^1, q^2 \in O_{II}(A)$. Then $q^1, q^2 \in S^c$, and therefore $q^1, q^2 \in O_{II}^c(A)$. So $q^1 = q^2$; $q \in ext(O_{II}(A))$. $\|$

<u>LEMMA 2.</u> $O_{II}^c(A) = ch(ext(O_{II}^c(A)))$.

<u>PROOF</u>. Let $q = (q_1, q_2, \dots) \in O_{II}^c(A)$. Take an $n \in \mathbb{N}$ such that $q_j = 0$ for each $j > n$. Let $\widetilde{A}$ be the $m \times n$-matrix $[a_{ij}]_{i=1,j=1}^{m\ \ n}$, $\widetilde{q} := (q_1, q_2, \dots, q_n)$ and $P := \{y \in S^n;\ \widetilde{A}y^t \leq (v(A), v(A), \dots, v(A))^t\}$. Then $\widetilde{q} \in P$. Since P is a compact convex subset of $\mathbb{R}^n$, it follows that $\widetilde{q} \in ch(ext(P))$. Let $z^{(1)}, z^{(2)}, \dots, z^{(r)}$ be extreme points of P and $(p_1, \dots, p_r) \in S^r$ such that $\widetilde{q} = \Sigma_{k=1}^r p_k z^{(k)}$. Put $w^{(k)} := (z_1^{(k)}, \dots, z_n^{(k)}, 0, 0, \dots) \in l_c^1$ for each $k \in \{1,2,\dots,r\}$. Then $w^{(k)} \in ext(O_{II}^c(A))$ and $q = \Sigma_{k=1}^r p_k w^{(k)} \in ch(ext(O_{II}^c(A))$. Hence $O_{II}^c(A) \subset ch(ext(O_{II}^c(A))$. Trivially, $ch(ext(O_{II}^c(A)))$ is a subset of the convex set $O_{II}^c(A)$. $\|$

Combining the theorems 3 and 4 and lemma 2 we obtain a proof of

THEOREM 5. $O_{II}(A)$ *is a closed convex subset of* l^1. *Moreover,*

$$O_{II}(A) = cl(ch(ext(O_{II}(A)))).$$

REMARKS.

1. The following examples show that $O_{II}(A)$ may be empty or non-empty.
   (a) For the $1\times\infty$-matrix $A = [1 \ \ \frac{1}{2} \ \ \frac{1}{3} \ \ldots]$ we have $v(A) = 0$,
   $O_I(A) = \{e_1\}$ and $O_{II}(A) = \emptyset$.
   (b) For the $1\times\infty$-matrix $A = [0 \ \ 0 \ \ 0 \ \ldots]$ we have $v(A) = 0$,
   $O_I(A) = \{e_1\}$ and $O_{II}(A) = S^\infty$.
   Note that $O_{II}(A)$ is not a compact set in example b. (For A with compact $O_{II}(A)$ theorem 3 would be a direct consequence of Krein-Milman's theorem.)
2. An alternative proof of theorem 5 was suggested to me by J. Borwein and goes as follows. First show that $O_{II}(A)$ is a closed convex subset of l^1. (This can be done in a way similar to the proof of theorem 1.) Since $O_{II}(A)$ is also bounded, then use can be made of theorem 1 in paper [6] of J. Lindenstrauss.

In view of theorem 5 it is interesting to study the extreme points of $O_{II}(A)$. In the following theorem we associate these extreme points with certain (finite) square submatrices of A.

THEOREM 6. *Let* A *be a bounded* $m\times\infty$*-matrix such that* $v(A) \neq 0$. *Let* $q \in O^c_{II}(A)$. *Then the following two assertions are equivalent.*
(1) $q \in ext(O^c_{II}(A))$.
(2) *There exists a non-singular square submatrix* K *of* A *such that*

$$q^t_K = v(A)\, K^{-1}\, \underline{1}^t \quad \textit{and} \quad v(A) = (\underline{1}\, K^{-1}\, \underline{1}^t)^{-1}.$$

[Here q_K is the vector obtained from q by removing the coordinates corresponding to the columns of A which play no role in K; $\underline{1}$ is the vector for which all coordinates are equal to one and where the number of coordinates is equal to the number of rows of K.]

PROOF. (a) We want to show that (1) implies (2). Let $q \in ext(O^c_{II}(A))$ and let $C := \{j \in \mathbb{N}\, ;\, q_j > 0\}$. Without loss of generality we suppose that $C = \{1,2,\ldots,n\}$. (Otherwise, rearrange the columns). Let $\tilde{A} := [a_{ij}]^{m}_{i=1,j=1}{}^{n}$ and $\tilde{q} := (q_1,\ldots,q_n)$.

(a.1) Let $M := \{e_i\tilde{A};\ i \in \{1,\dots,m\},\ e_i\tilde{A}\,\tilde{q}^t = v(A)\}$. We note that for each $p \in O_I(A)$ the following holds: if $p_i > 0$, then $e_i\tilde{A}\,\tilde{q}^t = v(A)$. Since $O_I(A) \neq \emptyset$ by theorem 1, we may conclude that $M \neq \emptyset$.

(a.2) Now we prove that $(1,1,\dots,1) \in \mathbb{R}^n$ is an element of the linear hull of M. Take $p \in O_I(A)$. It follows from $p\tilde{A}\,\tilde{q}^t = pA\,q^t = v(A)$, $p\tilde{A} \geq v(A)(1,1,\dots,1)$, $\tilde{q}_j > 0$ for each $j \in \{1,\dots,n\}$ and $\Sigma_{j=1}^n \tilde{q}_j = 1$, that $p\tilde{A} = v(A)(1,1,\dots,1)$. So $(1,1,\dots,1)$ is a linear combination of the rows of $\tilde{A}$ in $M(v(A) \neq 0)$.

(a.3) From $q \in ext(O_{II}^c(A))$ we may conclude that $\tilde{q}$ is an extreme point of the polyhedral subset

$$P := \{y \in \mathbb{R}^n;\ \tilde{A}y \leq (v(A),\dots,v(A))^t,\ y \geq 0,\ (1,1,\dots,1)y^t = 1\}$$

of $\mathbb{R}^n$.

(a.4) It is well known (cf. [3], p.74, Satz B) that $\tilde{q} \in ext(P)$ and $\tilde{q}_j > 0$ for each $j \in \{1,\dots,n\}$ imply that $M \cup \{(1,1,\dots,1)\}$ contains a basis of $\mathbb{R}^n$.

(a.5) In view of (a.2) and (a.4) we can find n linearly independent rows of $\tilde{A}$ in M. Let K be the n×n-matrix consisting of these rows. Then it is obvious that K is a non-singular submatrix of A and that $q_K = \tilde{q}$. Further $Kq_K^t = K\tilde{q}^t = v(A)\,\underline{1}^t$, so $q_K^t = v(A)K^{-1}\underline{1}^t$; and $1 = \Sigma_{j=1}^n \tilde{q}_j = \underline{1}q_K^t = v(A)\underline{1}K^{-1}\underline{1}^t$, so $v(A) = (\underline{1}\ K^{-1}\underline{1}^t)^{-1}$.

(b) Suppose now that (2) holds. Let $q^1, q^2 \in O_{II}^c(A)$ such that $q = \frac{1}{2}(q^1+q^2)$. Then $(q^1)_j = (q^2)_j = q_j = 0$ for each j which corresponds with a column of A which does not play a role in K (q_K is a probability vector). From the non-singularity of K and

$$Kq_K^t = v(A)\underline{1}^t,\ K(q_K^1)^t \leq v(A)\underline{1}^t,\ K(q_K^2)^t \leq v(A)\underline{1}^t$$

we can conclude that $q_j = (q^1)_j = (q^2)_j$ for each j which corresponds with a column of K. Hence $q^1 = q^2$, $q \in ext(O_{II}^c(A))$, and we have proved that (2) implies (1). ||

Using the same techniques as in [8], pp.33-34 it can be seen that the following theorem is a corollary of theorem 6. (Here the adjoint matrix of a square matrix K is denoted by adj(K).)

<u>THEOREM 7</u>. *Let A be a bounded m×∞-matrix. Let $q \in O_{II}^c(A)$. Then the*

following two assertions are equivalent.

(1) $q \in \mathrm{ext}(O^c_{II}(A))$.

(2) *There exists a square submatrix* K *of* A *such that*
$\alpha := \underline{1}\ \mathrm{adj}(K)\ \underline{1}^t \neq 0$, $v(A) = \alpha^{-1}\det(K)$, $q_K = \alpha^{-1}\mathrm{adj}(K)\underline{1}^t$.

For $q \in l^1_c$ we shall call the set $C(q) := \{j \in \mathbb{N} : q_j \neq 0\}$ the *carrier* of q. It follows from theorem 7 that extreme points of $O_{II}(A)$ have a carrier with at most m elements (the size of a square submatrix K of the $m\times\infty$-matrix A is at most $m\times m$). Particularly, we may conclude that there is at least one optimal strategy q with $|C(q)| \le m$ if $O^c_{II}(A) \neq \emptyset$. Furthermore, we note that we may conclude with the aid of theorem 7 that $O^c_{II}(A)$ is a polytope (i.e. the convex hull of a finite set) iff there is an $n \in \mathbb{N}$ such that $C(q) \subset \{1,2,\dots,n\}$ for each $q \in O^c_{II}(A)$. [The number of square submatrices of the $m\times n$-matrix $[a_{ij}]^{m}_{i=1,}{}^{n}_{j=1}$ is finite.] It is well-known (Cf. [8]) that for each finite matrix game A the two solution sets $O_I(A)$ and $O_{II}(A)$ are polytopes. For a more detailed description of $O^c_{II}(A)$ for an $m\times\infty$-matrix A we need some definitions.

(1) Let Y be a non-empty subset of S^c. Suppose that there exists a countable subset T of l^1_c such that $Y = \mathrm{ch}(T)$ and such that for each $n \in \mathbb{N}$ the set $T_n := \{z \in T;\ C(z) \subset \{1,2,\dots,n\}\}$ is a finite set. Then we shall call Y a *quasi-polytope*.
Let $Y_n := \mathrm{ch}(T_n)$ for each $n \in \mathbb{N}$. Let $f_n := 0$ if $Y_n = \emptyset$, and let f_n be the number of interior bounding faces (Cf. [4] p.42) of $\tilde{Y}_n := \{(x_1,\dots,x_n) \in S^n;\ (x_1,\dots,x_n,0,0,\dots) \in Y_n\}$ if $Y_n \neq \emptyset$. Then $f := \sup\{f_n;\ n \in \mathbb{N}\}$ is called the *face-number* of Y. ($f \in \mathbb{N} \cup \{0,\infty\}$).

(2) A linear subspace V of l^1_c is called a *co-finite subspace* if there exist a $k \in \mathbb{N}$ and $x^1,x^2,\dots,x^k \in l^\infty$ such that $V = \{y \in l^1_c;\ x^i y^t = 0$ for each $i \in \{1,\dots,k\}\}$. (l^∞ is the set of bounded infinite sequences of real numbers.)

<u>THEOREM 8</u>. *Suppose that* $O^c_{II}(A) \neq \emptyset$ *and that* $v(A) = 0$. *Then*

(1) $O^c_{II}(A)$ *is a quasi-polytope*

(2) $O^c_{II}(A)$ *has a finite face-number*

(3) *There exists a co-finite subspace* V *of* l^1_c *such that*
$\mathrm{lh}(O^c_{II}(A)) \cap S^c = V \cap S^c$.

[$lh(O^c_{II}(A))$ *is the linear hull of* $O^c_{II}(A)$.]

PROOF. (1) Let T_n be the set of those extreme points q of $O^c_{II}(A)$ such that $C(q) \subset \{1,\dots,n\}$ $(n \in \mathbb{N})$, and $T := \bigcup_{n\in\mathbb{N}} T_n$. Then $T = \mathrm{ext}(O^c_{II}(A))$, and $ch(T) = O^c_{II}(A)$ by lemma 2. In view of theorem 7 each element of T_n corresponds with at least one square submatrix of $A_n := [a_{ij}]^{m}_{i=1,}{}^{n}_{j=1}$ and it is obvious that with each square submatrix K of A_n there corresponds at most one element of T_n. Since there are only a finite number of square submatrices of A_n, we conclude that T_n is finite. Then T is a finite or a countably infinite set and thus $O^c_{II}(A)$ is a quasi-polytope.

(2) Note that $Y_n := ch(T_n) \neq \emptyset$ iff $v(A_n) = v(A)$, $O_{II}(A_n) = \tilde{Y}_n$. If $Y_n = \emptyset$, then $f_n = 0$. Because $O^c_{II}(A) \neq \emptyset$, for n sufficiently large, $Y_n \neq \emptyset$ and then $f_n < m$ in view of (2) of the principal theorem in [4] p.42. Hence $f < m$; the face-number of $O^c_{II}(A)$ is finite.

(3) Let $G := \{i \in \{1,\dots,m\};\ e_iAq^t = v(A)$ for each $q \in O^c_{II}(A)\}$. Note that $G \neq \emptyset$. (If $p \in O_I(A) \neq \emptyset$ and $p_i > 0$, then $i \in G$.) Without loss of generality we suppose that $G = \{1,2,\dots,k\}$. Then

$$O^c_{II}(A) = S^c \cap \bigcap_{i=1}^{k} \{y \in l^1_c;\ e_iAy^t = 0\} \cap \bigcap_{i=k+1}^{m} \{y \in l^1_c;\ e_iAy^t \leq 0\}$$

because $v(A) = 0$.

Let $V := \bigcap_{i=1}^{k} \{y \in l^1_c;\ e_iAy^t = 0\}$. Then V is a co-finite linear subspace of l^1_c. $O^c_{II}(A) \subset V \cap S^c$ implies that $lh(O^c_{II}(A)) \cap S^c \subset V \cap S^c$.

Let $z \in V \cap S^c$. We shall prove that $z \in lh(O^c_{II}(A)) \cap S^c$.

This is true if $k = m$. Suppose that $k < m$. For each $i \in \{k+1,\dots,m\}$ take a $y^i \in O^c_{II}(A)$ such that $e_iA(y^i)^t < 0$ and let $\hat{y} := (m-k)^{-1} \Sigma^m_{i=k+1} y^i \in O^c_{II}(A)$. For each $t \in (0,1)$ and each $i \in \{1,\dots,k\}$ we have $e_iA(t\hat{y}+(1-t)z)^t = 0$. We may take a $t_0 \in (0,1)$ such that $e_iA(t_0\hat{y}+(1-t_0)z)^t \leq 0$ for each $i \in \{k+1,\dots,m\}$ because $e_iA\hat{y}^t < 0$ for each $i \in \{k+1,\dots,m\}$. So $t_0\hat{y}+(1-t_0)z \in O^c_{II}(A)$. But then $z \in lh(O^c_{II}(A))$. Hence $V \cap S^c = lh(O^c_{II}(A)) \cap S^c$. ||

In [9] pp.57-60 it is proved that the following converse of theorem 8 also holds: let Y be a quasi-polytope in S^c with a finite face-number and suppose that there exists a co-finite subspace V of l^1_c such that

$lh(Y) \cap S^c = V \cap S^c$. Then there is an $m \in \mathbb{N}$ and an $m \times \infty$-matrix A such that $v(A) = 0$ and $O^c_{II}(A) = Y$.

In [9] much attention is paid to the problem of the construction of semi-infinite matrix games with prescribed optimal strategy spaces.

As a final remark we note that we restricted our attention in this paper to bounded semi-infinite matrix games to avoid technical complications. Some of our results can be extended to unbounded matrix games.

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